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The measure of the interior angle of a regular convex polygon with n sides is an integer θ.

The sum of the exterior angles of a polygon is 360 degrees. An exterior angle and interior angle at the same vertex form a straight angle.

$$360 = n(180 - \theta)$$
$$360 + n\theta = 180n$$
$$n\theta = 180n - 360$$
$$\theta = 180 - \frac{360}{n}$$

$\frac{360}{n}$ is an integer in the domain $[1, 120]$ because θ is an integer, $n \geq 3$ because all polygons have three or more sides, and $\frac{360}{n} > 0$ because θ, the interior angle, must be less than 180 for the polygon to be convex.

Values of n must be factors of 360 in the domain $[3, 360]$, as positive factors of positive integers are less than or equal to their products and all polygons have at least three sides. The prime factorization of 360 is:

$$2 \cdot 2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 = 360$$

Values of n are all products of the prime factors of 360. One could find all distinct factors within the domain using combinations and permutations, but in this case trial and error is just as easy.

$$3, 4, 5, 6, 8, 9, 10, 12, 15, 18, 20, 24, 30, 36, 40, 45, 60, 72, 90, 120, 180, 360$$

There are 22 integers n representing polygons fulfilling the requirements of the problem.

$$\sqrt{a}+\sqrt{b}+\sqrt{c}=\sqrt{219+\sqrt{10080}+\sqrt{12600}+\sqrt{35280}}$$
$$(\sqrt{a}+\sqrt{b}+\sqrt{c})^2=219+\sqrt{10080}+\sqrt{12600}+\sqrt{35280}$$
$$a+b+c+2\sqrt{ab}+2\sqrt{bc}+2\sqrt{ac}=219+\sqrt{10080}+\sqrt{12600}+\sqrt{35280}$$

$$a+b+c=219$$
$$2\sqrt{ab}=\sqrt{10080}$$
$$2\sqrt{bc}=\sqrt{12600}$$
$$2\sqrt{ac}=\sqrt{35280}$$

$$ab=2520=2^3\cdot 3^2\cdot 5\cdot 7$$
$$bc=3150=2\cdot 3^2\cdot 5^2\cdot 7$$
$$ac=8820=2^2\cdot 3^2\cdot 5\cdot 7^2$$

$$a=2^2\cdot 3\cdot 7=84$$
$$b=2\cdot 3\cdot 5=30$$
$$c=3\cdot 5\cdot 7=105$$

Substitution into line 1:

$$\sqrt{84}+\sqrt{30}+\sqrt{105}=\sqrt{219+\sqrt{10080}+\sqrt{12600}+\sqrt{35280}}$$

$$84+30+105+2\sqrt{84\cdot 30}+2\sqrt{30\cdot 105}+2\sqrt{84\cdot 105}=219+\sqrt{10080}+\sqrt{12600}+\sqrt{35280}$$

$$219+\sqrt{10080}+\sqrt{12600}+\sqrt{35280}=219+\sqrt{10080}+\sqrt{12600}+\sqrt{35280}$$

Case 1:
Let y, z be 0.

$$3x = 23$$

$$2x^2 = n_1$$

$$2x^2 + 3x - 23 = n_1$$

$$\frac{dn_1}{dx} = 4x + 2$$

$$4x + 2 = 0$$

$$x = -\frac{1}{2}$$

n_1 has a minimum at $(-\frac{1}{2}, -24)$.

Case 2:
Let x, z be 0.
n_2 has a minimum at $(-\frac{2}{3}, -\frac{72}{3})$.

Case 3:
Let x, y be 0.
n_3 has a minimum at $(-\frac{5}{12}, -\frac{577}{24})$.

n cannot be less than the sum of these minimums, which is $-\frac{1729}{24} \approx -72$.

Definition of median:

$$\overline{CD} \cong \overline{DB}$$

$$\overline{DG} \cong \overline{DG}$$

The circle inscribed in $\triangle CDG$ has a center H. There is a segment $\overline{HJ}$ where $\overline{CD}$ is tangent at J to circle H, and a segment $\overline{KH}$, where $\overline{DG}$ is tangent to H at point K.

$\overline{HJ} \perp \overline{CD} \wedge \overline{KH} \perp \overline{DK}$ by definitions of radius (normal line of a circle). $DJHK$ is a kite with two opposite right angles at K and J.

The circle inscribed in $\triangle BDG$ has a center I. There is a segment $\overline{IL}$ where $\overline{BD}$ is tangent at L to circle I, and a segment $\overline{IM}$, where $\overline{DG}$ is tangent to I at point M.

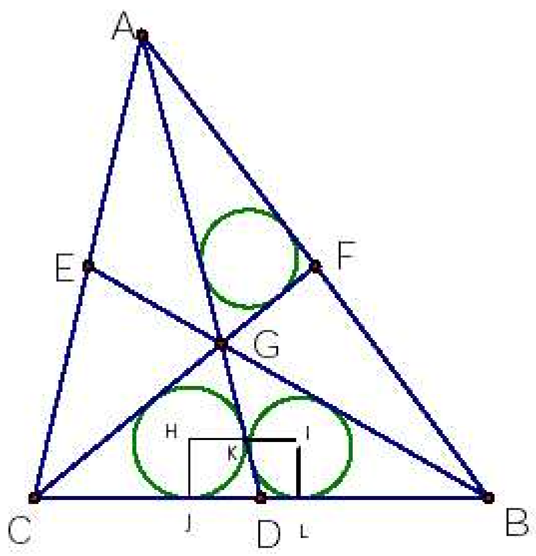
$\overline{IL} \perp \overline{DB} \wedge \overline{MI} \perp \overline{DM}$ by definitions of radius (normal line of a circle). $DLIM$ is a kite with two opposite right angles at M and L.

$\overline{JH} \parallel \overline{IL}$ by supplementary adjacent interior angles. $\overline{IM} \parallel \overline{HK}$ by various transverse line theorems. Congruent circles have congruent radii, so K is identical to M.

$\triangle ADC$ and $\triangle ADB$ are right triangles with congruent bases $\overline{CD}$ and $\overline{DB}$, identity side $\overline{AD}$, and congruent angles $\angle CDA$ and $\angle BDA$.

$$\triangle ADC \cong \triangle ADB$$

$\triangle ABC$ is isosceles.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