A Question About Tiles

I feel it is reasonable to assume that the tiles may not overlap, as the young woman would either trip on the rough surface made by overlapping tiles or be forced to cut them.

The total area of the tiles is $6 \cdot 6 \cdot 24 = 864in^2$. The absolute area of the circle covered by the tiles, in square inches, is $A \leq 864$. The radius of the circle is r. Using $A = \pi r^2$, an equation can be given for the circle so that its center is at the origin.

$$x^2 + y^2 = \frac{A}{\pi}$$

$$x^2 + y^2 \leq \frac{864}{\pi}$$

$$y = \pm\sqrt{-x^2 + \frac{A}{\pi}}$$

$$y \leq \sqrt{-x^2 + \frac{864}{\pi}} \quad \wedge \quad y \geq -\sqrt{-x^2 + \frac{864}{\pi}}$$

Any tiling pattern has an efficiency $E = \frac{A}{864}$.

If the tiles were simply arranged in a 6×4 rectangle, the circle inscribed in the rectangle would have a radius

$$r = 12 \quad \rightarrow \quad A = 144\pi$$

$$E \approx .52$$

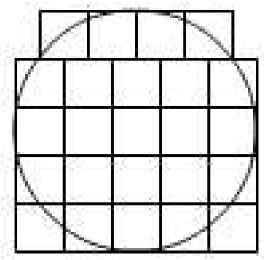
If a somewhat better arrangement were used, with a 5×4 rectangle and the four remaining tiles placed along a long side, offset from the corners by 3".

$$r = \frac{5 \cdot 6}{2} = 15 \quad \rightarrow \quad A = 225\pi$$

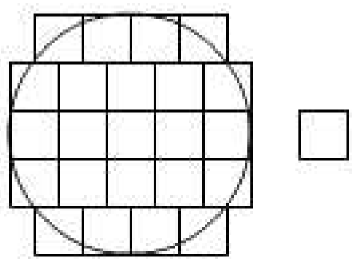
$$y = \sqrt{-x^2 + 225} \quad \wedge \quad y = -\sqrt{-x^2 + 225}$$

$$E \approx .82$$

This method uses the tiles with about 82% efficiency. About 4.4 tiles are wasted. For comparison, a circle inscribed in a perfect square, the ideal arrangment of a single tile, is only 79% efficient. If the center of the circle is at the origin, it happens that the points (0,15), (0,-15), (15,0), (-15,0), (12,9), and (-12,9) are all 15 from the origin, so that the circle touches the perimeter of the tiles at six places. One of the tiles in the bottom row could be removed, and the others slid over, to make the pattern more symmetric without decreasing the area of the circle. Due to the even distribution of touch points, increased to eight, the extra tile would not be useful.



Sketch 1



Sketch 2

This method seems to be the most efficient.

A variation on Riemann Sums can be used to find r if half the tiles are arranged over a semicircle, each tile in a column starting at the straight edge of the semicircle.

Definition: If $f(x) = \lceil x \rceil$ then f is the smallest integer greater than or equal to x. This is known as the ceiling or least integer function[1].

[1]I happened to discover this integral while examining this problem. $\int \lceil x \rceil dx = \lceil x \rceil \left(x - \frac{\lfloor x \rfloor}{2} \right) + C$

Definition: If $f(x) = \lfloor x \rfloor$ then f is the largest integer less than or equal to x This is known as the floor or greatest integer function.

I used these functions to change the Riemann Sum so that the approximating rectangles were 6" wide and a multiple of 6" tall.

n is the number of columns formed by the tiles on one side of the semicircle.

For an odd number of columns:

$$n = \left\lfloor \frac{r}{6} \right\rfloor$$

$$2\sum_{i=1}^{n} 6\left(6\left\lceil \frac{\sqrt{-\left(6i-r\right)^2+r^2}}{6} \right\rceil\right) + 36\left\lceil \frac{r}{6} \right\rceil \leq \frac{864}{2}$$

$$2\sum_{i=1}^{\left\lfloor \frac{r}{6} \right\rfloor} 6\left(6\left\lceil \frac{\sqrt{-\left(6i-r\right)^2+r^2}}{6} \right\rceil\right) + 36\left\lceil \frac{r}{6} \right\rceil \leq \frac{864}{2}$$

$$2\sum_{i=1}^{\left\lfloor \frac{r}{6} \right\rfloor} \left(\left\lceil \frac{\sqrt{12ri-36i^2}}{6} \right\rceil\right) + \left\lceil \frac{r}{6} \right\rceil \leq 12$$

For an even number of columns:

$$n = \left\lceil \frac{r}{6} \right\rceil$$

$$2\sum_{i=1}^{n} 6\left(6\left\lceil \frac{\sqrt{-\left(6i-6\right)^2+r^2}}{6} \right\rceil\right) \leq \frac{864}{2}$$

$$2\sum_{i=1}^{\left\lceil \frac{r}{6} \right\rceil} 6\left(6\left\lceil \frac{\sqrt{-\left(6i-6\right)^2+r^2}}{6} \right\rceil\right) \leq \frac{864}{2}$$

$$\sum_{i=1}^{\left\lceil \frac{r}{6} \right\rceil} \left(\left\lceil \frac{\sqrt{12ri-36i^2}}{6} \right\rceil\right) \leq 6$$

I have put a lot of consideration into these equations, but I am far from certain if they are accurate because of the large number of errors I have already found. They have three major flaws that I am aware of: First, there is no way to determine if the equation for an odd number of columns or even number of columns is better without evaluating them. The two sets of equations are needed to deal with the possibility of a tile that touches the inscribed circle along the top or bottom edge instead of at a vertex. If the wrong one is used, the result could be an incompletely covered circle or a circle where a whole column of tiles is wasted. Second, I have failed to find an efficient way to evaluate the sum of a series of ceiling or floor functions. I have used trial and error to find r. Third, a straight edge is forced along the bottom of the semicircles. I believe this difficulty can be resolved by doubling the y part of the relation and the number of tiles used in the region considered (the right side of the equals sign). I created this straight edge because it made the problem more like a familiar Riemann Sum.

Odd:

$$2\sum_{i=1}^{\lfloor \frac{r}{6} \rfloor}\left(\left\lceil \frac{\mathbf{2}\cdot\sqrt{12ri-36i^2}}{6} \right\rceil\right)+\left\lceil \frac{\mathbf{2}r}{6} \right\rceil \leq \mathbf{24}$$

$$r \leq 15$$

$r = 15$ has 5 columns and does not leave a gap, but it does not use 1 tile. In fact, it is identical to sketch 2.

Even:

$$\sum_{i=1}^{\lceil \frac{r}{6} \rceil}\left(\left\lceil \frac{\mathbf{2}\cdot\sqrt{12ri-36i^2}}{6} \right\rceil\right) \leq \mathbf{12}$$

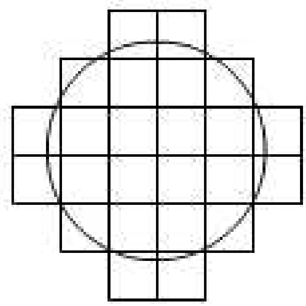
$$r \leq 12$$

$r = 12$ is a 4×4 rectangle. It does not use 4 tiles. Clearly this scenario requires an odd number of columns.

Another attempt at tiling:

$A = 180\pi$

$E \approx .65$



Sketch 3

I propose a general method of optimizing n congruent rectangular tiles to allow the inscription a circle with the largest possible area and a center at the origin:

1. Arrange the tiles into a horizontal row.
2. Draw the largest possible circle that can fit in the tiles.
3. Translate the circle as far as possible to the left without removing it from the tiles.
4. Remove one tile on the right that is not covering part of the circle, and place it on the lowest point in the top half of the pattern where the outer boundary of the tiling pattern perimeter meets the circle's edge.
5. Translate each row of tiles so that the center of the row falls on the y axis.
6. Repeat 2-5 translating the circle to the right, removing tiles from the left, and placing them on the bottom.
7. Repeat 2-6 until no more tiles can be moved because the circle must cover part of each.